

High- T_c superconductivity of electron systems with flat bands pinned to the Fermi surface

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The phenomenon of flat bands pinned to the Fermi surface is analyzed on the basis of the Landau-Pitaevskii relation, which is applicable to electron systems of solids. It is shown that the gross properties of normal states of high- T_c superconductors, frequently called strange metals, are adequately explained within the flat-band scenario. Most notably, we demonstrate that in electron systems moving in a two-dimensional Brillouin zone, superconductivity may exist in domains of the Lifshitz phase diagram lying far from lines of critical antiferromagnetic fluctuations, even if the effective electron-electron interaction in the Cooper channel is repulsive.

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A central issue confronting present-day condensed matter theory is that of unambiguous determination of the mechanisms governing the rich non-Fermi-liquid (NFL) behavior revealed by intensive experimental studies of strongly correlated electron systems of solids and liquid ^3He films. Prominent on the scene of this seminal area of condensed matter physics are numerous versions of the Hertz-Millis-Moriya (HMM) description. These scenarios ascribe such NFL behavior to quantum critical fluctuations. Proponents of the HMM approach claim that accounting for such fluctuations also allows one to explain the phenomenon of high- T_c superconductivity, which entails substantial enhancement of the critical temperatures T_c of superconducting phase transitions relative to standard BCS values.

In BCS theory, a typical $T - x$ phase diagram of a superconducting system (x being a relevant control parameter such as doping or pressure) consists of an "island" of superconductivity, surrounded by a Fermi-liquid (FL) "sea," a domain where the resistivity varies as $\rho_{FL}(T) = \rho_0 + A_2 T^2$. Real phase diagrams of strongly correlated metals exhibiting superconductivity look different, as exemplified by the phase diagram of the LCCO family of electron-doped high- T_c superconductors. This phase diagram, established empirically in Refs. [1, 2], is reproduced in Fig. 1. A prominent feature is the presence of two distinct regimes of NFL temperature behavior of the resistivity $\rho(T)$ at $T > T_c$, which separate the "blue FL sea" from the "yellow island" of superconductivity. In the interval $T_c(x) < T < T_1(x)$, the resistivity $\rho(T)$ changes linearly with T , thus $\rho(T) = \rho_0 + A_1 T$. Above $T_1(x)$ a different NFL regime appears, in which $\rho(T)$ varies as T^n with $n \simeq 1.6$.

In the conventional HMM approach, the NFL linearity of the resistivity $\rho(T)$ observed in normal states of many

high- T_c compounds is attributed to antiferromagnetic critical fluctuations with wave vector $\mathbf{Q} = (\pi/a, \pi/a)$. In two-dimensional (2D) systems of electrons moving in the external field of a square lattice, this explanation works provided saddle points are located close to the Fermi lines; otherwise it *fails*. Significantly, the additional NFL regime in the LCCO phase diagram of Fig. 1 having $n \simeq 1.6$ in $\rho(T) = \rho_0 + A_n T^n$ terminates at the same critical doping as the linear-in- T regime. This behavior cannot plausibly be associated with antiferromagnetic fluctuations [3]. Further, there are numerous examples of 3D systems whose low- T resistivity also varies linearly with T , in contrast to predictions of the spin-fluctuation scenario for 3D systems. In addition, fluctuation-induced phenomena such as critical opalescence, exhibited as a huge enhancement in absorption of light by a liquid that is ordinarily transparent, emerge only in the *immediate vicinity* of points (or lines) of second-order phase transitions. The range ΔT of the interval $T_N - T$ impacted by critical fluctuations on the ordered side of the transition is determined by equating the mean-field value of the order parameter, behaving as $\sqrt{T_N - T}$, to the corresponding fluctuation contribution. Consequently, the fluctuation scenarios become irrelevant when $|T - T_N| > \Delta T$.

In fact, experiments on the systems of interest often furnish clear evidence for the persistence of NFL behavior *far* from such lines of criticality. In the well-studied heavy-fermion metal YbRh_2Si_2 [4], which provides the classic example of NFL linear behavior of $\rho(T)$ in the disordered phase *and* a T^2 resistivity regime on the ordered side of the posited antiferromagnetic phase transition, the latter behavior is found to prevail *almost up to the transition temperature* $T_N = 70$ mK. Given that the FL regime of $\rho(T)$ holds in the antiferromagnetic state of YbRh_2Si_2 almost up to T_N , the critical fluctuations must

be weak; otherwise the linear-in- T corrections to $\rho_{FL}(T)$ due to these fluctuations would be significant on the ordered side as well. However, their weakness stands in blatant contradiction to the linear-in- T variation of the resistivity on the disordered side, which is maintained from $T_N = 70\text{ mK}$ up to $T > 1\text{ K}$ [4]. The inescapable conclusion is that the proposed fluctuation mechanism is irrelevant to the NFL behavior of $\rho(T > T_N)$ in this compound.

This example is not alone. There are multiple instances of other materials in which the FL behavior of $\rho(T)$ persists solely on the ordered side of the transition, whereas on the disordered side the resistivity changes linearly with T – despite the expectation that FL theory should be *more* secure there than in the ordered state. To be specific, consider the $T - P$ phase diagram of the strongly correlated heavy-fermion metal CeCoIn_5 [5], presented in Fig. 2. A remarkable feature seen in this diagram is the crossover from the NFL linear-in- T regime of $\rho(T)$ to its FL regime, occurring in the normal state far from the transition point. This process is accompanied by a dramatic change of the residual resistivity ρ_0 , which drops by factor around ten on the FL side of the crossover [5]. Such behavior is inconceivable within the textbook understanding of kinetic phenomena in Fermi liquids.

All these facts and many others portend that in some strongly correlated electron systems of solids, it is the single-particle degrees of freedom that are the real play-makers in the observed NFL behavior, rather than critical fluctuations. By implication the HHM approach is fallacious when applied to these systems, since the single-particle degrees are integrated out.

It has become clear that the distinctive signature of the underlying physics of the Lifshitz phase diagram is the appearance of a so-called quantum critical point (QCP) where the density of states $N(T = 0)$, associated in homogeneous matter with the effective mass M^* , is *divergent*. Accordingly, the Landau state becomes unstable beyond the QCP and necessarily undergoes rearrangement [6]. The relevance of the QCP to high- T_c superconductivity has been confirmed in recent experimental work of Ramshaw et al. [7] on $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$, one of the most prominent high- T_c superconductors, with T_c around 90 K. In documenting “A quantum critical point at the heart of high- T_c superconductivity,” they have studied the enhancement of the electron effective mass M^* in dHvA oscillations of the normal metallic state, subjecting the sample to huge magnetic fields B of more than 90 K that terminate superconductivity. The measured effective mass $M^*(B)$ is found to *diverge* at the optimal doping x_o , where the critical temperature $T_c(x, B = 0)$ attains its maximum.

It is noteworthy that the presence of a QCP in the Lifshitz phase diagram of homogeneous 3D electron systems was predicted in microscopic parameter-free calcu-

lations [8] prior to its experimental discovery in strongly correlated heavy-fermion metals [4]. Application of the approach of Ref. [8] to the 2D problem [9] demonstrated that in this case the QCP lies at realistic electron densities, corresponding to $r_s \simeq 8$.

Absent any change of symmetry, the anticipated rearrangement of the Landau state occurring at the QCP is naturally attributed to some topological transition. The earliest topological scenario for NFL behavior in strongly correlated Fermi systems, advanced more than 20 years ago [10–12], traced this behavior to an *interaction-induced* rearrangement of the Landau state, often called fermion condensation (FC). This phenomenon, described more vividly as a swelling of the Fermi surface, is associated with the occurrence of a flat band pinned to the Fermi surface.

In a significant formal development, the FC phenomenon was rediscovered in 2009 within the framework of the adS-CFT duality [13]. More to the point, the formation of flat bands has been demonstrated both analytically and numerically for the Hubbard model, one of the most popular models of strongly correlated electron systems [14]. Therefore the question of principle whether the FC rearrangement exists and is relevant to condensed matter theory already has a positive answer. The remaining issue, addressed herein and elsewhere, is whether or not the FC phenomenon can actually provide the basis for a satisfactory explanation of the experimentally observed NFL behavior of such systems. That the FC scenario competes favorably with other attempts to explain the salient experimental data has been established in many studies, notably Refs. [15–18]. Additionally, invocation of FC theory has recently resolved a long-standing puzzle associated with the disappearance of a specific set of Shubnikov-de Haas magnetic oscillations in the 2D electron gas of MOSFETs, which results in the doubling of oscillation periods near a quantum critical point [19, 20].

The paramount objective of this communication is to apply the flat-band scenario to the elucidation of high- T_c superconductivity, discovered 30 years ago and still a challenge to theoretical understanding. The adequacy of any theory of this phenomenon depends largely on how well it reproduces the properties of strange metals – normal states of high- T_c superconductors. The Landau FL theory of normal states provides the basis for BCS theory, which, however, fails to describe high- T_c superconductivity. With this in mind, we present the essential elements of FC theory, focusing on those departures from FL theory that may qualify it as a basis for understanding the phase diagrams of high- T_c materials.

It should be emphasized that FL and FC theories are both rooted in seminal work of Landau [21] in that they employ the quasiparticle formalism. Hence they are applicable provided the damping of single-particle excitations is small compared with their energy. In the Landau theory of conventional Fermi liquids, this require-

ment is always met toward $T = 0$, since the damping is proportional to T^2 . The situation is more complicated in the flat-band scenario, because systems having flat bands belong in fact to the class of marginal Fermi liquids, in which the damping changes linearly with T , but with a prefactor proportional to the ratio η of the volume occupied in momentum space by the flat bands to the total Fermi volume (see below). Thus, the above requirement for applicability is met provided η is small.

Importantly, in electron systems of solids where translational invariance breaks down, the single-particle states are identified by quasimomentum \mathbf{p} , and the FL relation

$$n(\mathbf{p}) = (1 + e^{\epsilon(\mathbf{p})/T})^{-1}, \quad (1)$$

between the quasiparticle momentum distribution $n(\mathbf{p})$ and the single-particle spectrum $\epsilon(\mathbf{p})$ (measured from the chemical potential μ), continues to apply. As shown in Ref. [22], the Landau-Pitaevskii (LP) identity can be employed as a second relation between these quantities that holds for the electron system moving in the external field of the crystal lattice. In the notation adopted here, the LP equation takes the form

$$\frac{\partial \epsilon(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial \epsilon_0(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n(\mathbf{p}')}{\partial \mathbf{p}'} dv', \quad (2)$$

where the quantity $\partial \epsilon_0(\mathbf{p})/\partial \mathbf{p}$ contains only regular contributions coming from domains far from the Fermi surface.

Like the corresponding set of equations for homogeneous matter the set (1) and (2) also possesses a class of flat-band solutions for which the electron group velocity vanishes in a finite domain $\mathbf{p} \in \Omega$. Accordingly, one is required to solve the reduced equation

$$0 = \frac{\partial \epsilon_0(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n_*(\mathbf{p}')}{\partial \mathbf{p}'} dv', \quad \mathbf{p}, \mathbf{p}' \in \Omega. \quad (3)$$

in this region. The function $n_*(\mathbf{p})$ is introduced to represent a nontrivial FC solution of this equation. Outside the FC region, the quasiparticle spectrum obeys the familiar Landau-type relation

$$\frac{\partial \epsilon(\mathbf{p})}{\partial \mathbf{p}} = \frac{\partial \epsilon_0(\mathbf{p})}{\partial \mathbf{p}} + \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n_*(\mathbf{p}')}{\partial \mathbf{p}'} dv', \quad \mathbf{p} \notin \Omega. \quad (4)$$

Such a solution is characterized by its topological charge (TC), an invariant expressed in terms of a contour integral constructed from the single-particle Green function $G(\mathbf{p}, \epsilon)$ and its derivatives [11, 24]. The TC of a state exhibiting a flat band takes a *half-odd-integral* value, whereas the TC assigned to a Lifshitz state featuring a multi-connected Fermi surface (or ‘‘Lifshitz pockets’’) is always integral, since this state has standard quasiparticle occupation numbers $n(\mathbf{p}) = 0, 1$.

Proceeding to low $T \neq 0$, one may insert the solution $n_*(\mathbf{p})$ of Eq. (3) as a zeroth approximation for $n(\mathbf{p}, T)$ to

obtain [12]

$$\epsilon(\mathbf{p}, T \rightarrow 0) = T \ln \frac{1 - n_*(\mathbf{p})}{n_*(\mathbf{p})}, \quad \mathbf{p} \in \Omega. \quad (5)$$

In the FC momentum region $\mathbf{p} \in \Omega$, the resulting dispersion $v_n = \partial \epsilon(\mathbf{p})/\partial p_n$ of the single-particle spectrum is then found to be proportional to T .

An essential feature of the flat-band scenario is the presence of a residual entropy [10, 17]

$$S_* = - \int_{\Omega} [(1 - n_*(\mathbf{p})) \ln(1 - n_*(\mathbf{p})) + n_*(\mathbf{p}) \ln n_*(\mathbf{p})] dv \quad (6)$$

associated with the FC region, where the occupation numbers $n_*(\mathbf{p})$ differ from 0 and 1. This residual entropy does not contribute at all to the specific heat $C(T) = T dS/dT$. However, in normal states of systems with flat bands, $S_*(T \rightarrow 0)$ retains a finite value and makes a huge T -independent contribution to the thermal expansion $\beta \propto \partial S/\partial P$ of these states. This conclusion is in agreement with the thermal expansion of the strongly correlated heavy-fermion metal CeCoIn₅ measured at $T > T_c \simeq 2.3$ K [25].

To avoid contradiction with the Nernst theorem mandating $S(0) = 0$, the residual entropy S_* must be released by means of some first- or second-order phase transitions, or with the aid of crossovers to a state having a multi-connected Fermi surface formed by a set of Lifshitz pockets with $n(\mathbf{p}) = 0, 1$, hence implying $S_* = 0$ [10, 17, 18]. This ramification explains the diversity of quantum phase transitions observed in strongly correlated electron systems, stemming from the interplay between antiferromagnetism, charge order, and superconductivity.

Let us examine more closely the NFL behavior of the resistivity $\rho(T)$ at $T > T_c$ in superconducting materials exhibiting flat bands, recognizing that this is the foundation of the phase diagrams presented in Figs. 1 and 2. Ideally, the collision term differs from zero only due to Umklapp processes. However, strongly correlated electron systems of solids have open Fermi surfaces where these processes work in full force, such that the detailed structure of the kernel can be ignored, and we are left with the integral

$$I(n) \propto \int [n_1 n_2 (1 - n'_1) (1 - n'_2) - n'_1 n'_2 (1 - n_1) (1 - n_2)] \\ \times \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2) \delta(\epsilon_1 + \epsilon_2 - \epsilon'_1 - \epsilon'_2) dv_1 dv'_1 dv_2 dv'_2 \quad (7)$$

Eq. (7) exhibits several factors $1/v_n(\mathbf{p}, T)$ upon making the standard replacement $d^3 p \rightarrow dS(d\epsilon/v_n(\mathbf{p}))$, where dS is an element of the isoenergetic surface. In conventional Fermi liquids, these factors are T -independent and yield the FL result. Contrariwise, in systems with flat bands, it is seen from Eq. (5) that the behavior $v_n(\mathbf{p}, T) \propto T$ applies in the ‘‘hot spot’’ associated with the corresponding FC region. Any momentum integration over this region

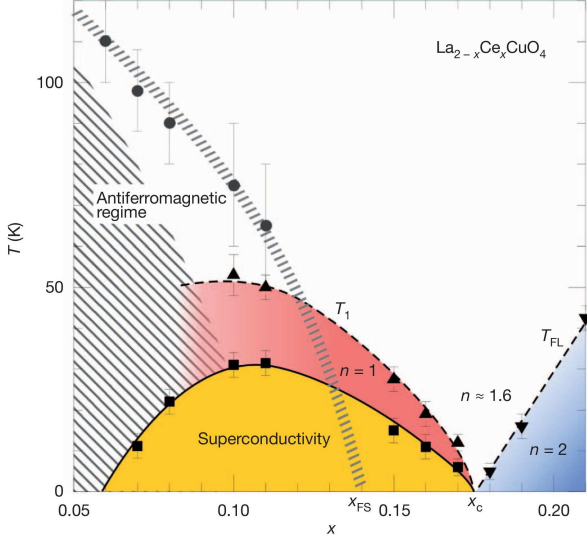


FIG. 1: (color online) Temperature-doping $T-x$ phase diagram of $\text{La}_{2-x}\text{Ce}_x\text{CuO}_4$ [1], reprinted with authors' permission. The yellow region is the superconducting dome. The resistivity $\rho(T)$ in the different normal phases has the form $\rho(T) = \rho_0 + A_n T^n$, with $n = 2$ for the FL domain (blue), $n = 1$ for the NFL FC domain (red), and $n = 1.6$ for the NFL QCP domain (white) (see the text). The temperatures T_1 (triangles) and T_{FL} (inverted triangles), traced by dashed curves, indicate the crossover temperatures to the linear-in- T regime of $\rho(T)$ and the FL regimes, respectively.

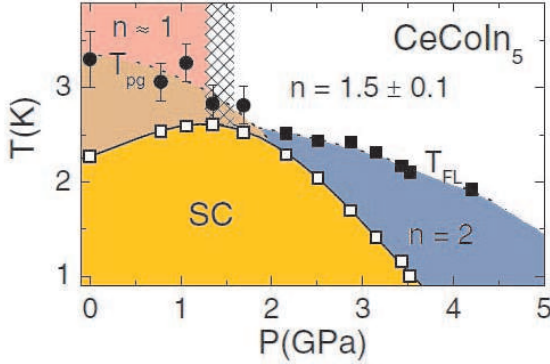


FIG. 2: (color online) Temperature-pressure $T-P$ phase diagram of CeCoIn_5 [5], constructed based on the phase diagram published in Ref. [5]. Unlike the phase diagram of Fig. 1, there exists a small pseudogap (PG) region in orange where superconductivity is precluded, but a gap Δ persists. The occurrence of a PG region in this compound will be discussed in detail in a future article.

then contributes a factor η/T to the collision integral (and to the resistivity as well), where η is the dimensionless FC density. This causes a damping of single-particle excitations, rendering systems with flat bands *marginal Fermi liquids* [26, 27]. Such a conclusion is in agreement

with the results of experimental studies of kinetic properties of the heavy-fermion metal CeCoIn_5 [5, 28].

Ordinarily η is very small, so we need only retain the leading terms of order η and η^2 to arrive at the resistivity expression

$$\rho(T) = \rho_0(P, x) + A_1(P, x)T \quad (8)$$

in the presence of a flat band, with

$$\rho_0(P, x) = \rho_0^i + a_0 \eta^2(P, x), \quad A_1(P, x) = a_1 \eta(P, x), \quad (9)$$

wherein ρ_0^i denotes an impurity-induced contribution to the residual resistivity ρ_0 and a_0, a_1 are constants. The total residual resistivity ρ_0 becomes dependent on pressure P and doping x . The cleaner the metal, the greater the magnitude of the jump of the residual resistivity ρ_0 in the rapid crossover from the NFL regime (9) to the standard FL regime. In the limit $\rho_0^i \rightarrow 0$, the ratio of the ρ_0 value on the NFL side of the crossover to that on the FL side tends to *infinity*. This observation serves to explain the rapid variation of ρ_0 found in the especially clean samples of CeCoIn_5 [5]. Such otherwise puzzling behavior has also been reported in measurements of the residual resistivity $\rho_0(P)$ of the metal CeAgSb_2 , where the aforesaid ratio attains huge values of order 10^2 [29].

As will be demonstrated below (cf. Eq. (15)), the critical temperature T_c for termination of superconductivity in high- T_c superconductors is proportional to the FC density η , so that

$$A_1(x) \propto T_c(x) \propto \eta(x). \quad (10)$$

Hence the theoretical ratio T_c/A_1 is independent of the FC density. While η changes somehow with doping x , the ratio T_c/A_1 turns out to be independent of x , in agreement with the experimental behavior uncovered in the electron-doped materials LCCO and PCCO, as well as in the Bechgaard class of organic superconductors $(\text{TMTSF})_2\text{PF}_6$ [1, 30]. In addition, the relation (10) implies that the factor A_1 , which specifies the linear-in- T NFL regime of resistivity, vanishes at the same doping x_c where high- T_c superconductivity terminates, again in agreement with experiment (see Fig. 1). Thus we infer that three different resistivity regimes come into play in the immediate vicinity of FC onset: (i) the FL regime $\rho(T) \propto T^2$, (ii) the FC regime $\rho(T) \propto T$, and (iii) the high- T_c superconducting regime with $\rho = 0$.

On the other hand, for heavy-fermion superconductors such as CeCoIn_5 in which the critical temperature T_c is extremely low, the BCS logarithmic term cannot be ignored. In this situation, the onset of FC is disconnected from the occurrence of superconductivity, and merging of different resistivity regimes in the corresponding normal states does not take place. This conclusion is seen to be in agreement with the $T-P$ phase diagram of CeCoIn_5 displayed in Fig. 2.

In the systems under study, still another profound NFL contribution to $\rho(T)$ is possible, specific to what can be called the QCP resistivity regime. Elucidation of its emergence is especially simple in the homogeneous electron liquid where, at the QCP, the Fermi velocity vanishes to yield

$$\epsilon(p \rightarrow p_F) \propto (p - p_F)|p - p_F| \quad (11)$$

and hence $v(p) = d\epsilon(p)/dp \propto \sqrt{|\epsilon(p)|}$. One easily verifies that the leading contribution to $\rho(T)$ now increases as $T^{3/2}$. This distinctive NFL regime has its onset at the critical doping x_c as well. It exists in those electron systems that possess a QCP, e.g. in the LCCO family [1] and in CeCoIn₅ [31]. There a minor difference between our theoretical predictions and experiment. In the white NFL regime of Fig. 1, the measured resistivity $\rho(T)$ varies as $T^{1.6}$, and in the corresponding regime of Fig. 2, as $\rho(T) \propto T^{1.5 \pm 0.1}$. Our analysis leads to the relation $\rho(T) = A_{3/2}T^{3/2} + A_2T^2$.

Having confirmed that the flat-band-scenario can successfully explain gross properties of the strange metals, we may now turn to the primary aim of this article: explanation of the occurrence of D -pairing and dramatic enhancement of its critical temperature T_c . In doing so we will proceed within the standard BCS theory, ignoring an enigmatic pseudogap phenomenon. In this case, the structure of the gap function and the magnitude of T_c are revealed with the aid of the linearized BCS gap equation

$$\Delta(\mathbf{p}) = - \int \mathcal{V}(\mathbf{p}, \mathbf{p}') \tanh\left(\frac{\epsilon(\mathbf{p}', T_c)}{2T_c}\right) \frac{\Delta(\mathbf{p}')}{2|\epsilon(\mathbf{p}', T_c)|} d\mathbf{v}'. \quad (12)$$

Upon defining the functions

$$X(\mathbf{p}, T_c) = \sqrt{\frac{\tanh(\epsilon(\mathbf{p}, T_c)/2T_c)}{\epsilon(\mathbf{p}, T_c)}},$$

$\zeta(\mathbf{p}) = \Delta(\mathbf{p})X(\mathbf{p}, T_c)$, and $H(\mathbf{p}, \mathbf{p}', T_c) = X(\mathbf{p}, T_c)X(\mathbf{p}', T_c)\mathcal{V}(\mathbf{p}, \mathbf{p}')$, Eq. (12) is conveniently rewritten in the form of a linear integral equation with a symmetric kernel,

$$\zeta(\mathbf{p}) = -\frac{1}{2} \int H(\mathbf{p}, \mathbf{p}', T_c) \zeta(\mathbf{p}') d\mathbf{v}'. \quad (13)$$

Employing Eq. (5)), we may obtain the result

$$X(\mathbf{p}, T_c) = T_c^{-1/2} \sqrt{\frac{1 - 2n_*(\mathbf{p})}{\ln((1 - n_*(\mathbf{p}))/n_*(\mathbf{p}))}}, \mathbf{p} \in \Omega \quad (14)$$

and observe that the function $X(\mathbf{p}, T_c)$ is greatly enhanced in FC domains.

In view of the inverse proportionality to T_c exhibited by the kernel H , we infer that the overwhelming contributions to the right side of Eq. (13) come from the FC

domains, provided the ratio η/T_c exceeds unity. In that case, solution of this equation is obviated, accounting for the minor change of the block \mathcal{V} in a FC region. Upon neglecting other contributions to Eq. (13) and introducing a reduced kernel $h(\mathbf{p}, \mathbf{p}') = (T_c/2)H(\mathbf{p}, \mathbf{p}')$, we are left with the simple equation

$$T_c \zeta(\mathbf{p}) = - \int_{\Omega} h(\mathbf{p}, \mathbf{p}') \zeta(\mathbf{p}') d\mathbf{v}', \quad (15)$$

in which the kernel h is practically T -independent. We thus arrive at the pivotal conclusion that T_c changes *linearly* with the strength of the interaction \mathcal{V} , a distinctive fingerprint of high- T_c superconductivity. Further, since the right side of Eq. (15) is, in fact, proportional to the FC volume, we affirm that T_c does vary linearly with the FC density η , in agreement with Eq. (10).

In what follows we address the case of a 2D quadratic Brillouin zone where a single small FC pocket resides in each quadrant of the zone. Eq. (15) then reduces to the set of algebraic equations

$$T_c \zeta_l = - \sum h_{lk} \zeta_k, \quad (16)$$

where we have introduced quantities $\zeta_l = \zeta(\mathbf{p}_l)$ and $h_{lk} = h(\mathbf{p}_l, \mathbf{p}_k)$, with indexes l, k running from 1 to 4. Obviously, $h_{ll} \equiv h_0$, $h_{l, l+1} \equiv h_1$, and $h_{l, l+2} \equiv h_2$ are l -independent. Their signs and magnitudes depend largely on the interplay between phonon attraction and Coulomb repulsion. To solve the system (16) it is advantageous to make the substitution $\zeta_k = e^{i\alpha k}$, with the requirement $e^{4i\alpha} = 1$. Analysis demonstrates that there are 4 different high- T_c solutions of the problem. The first solution, corresponding to $\alpha = 0$, where all ζ_k are the same, occurs provided $h_0 + 2h_1 + h_2 < 0$. It describes S -pairing, with respective critical temperature $T_c = -(h_0 + 2h_1 + h_2)$.

Another interesting solution, with

$$\alpha = \pi, \quad T_c = 2h_1 - h_0 - h_2, \quad (17)$$

has the usual D -pairing form, with $\zeta_1 = \zeta_3 = -\zeta_2 = -\zeta_4$. It occurs in the region of the Lifshitz phase diagram where $2h_1 - h_0 - h_2 > 0$. Near the line of critical antiferromagnetic fluctuations where conventionally $h_0 = h_2 = 0$ while $h_1 > 0$, the solution (17) coincides with the standard one. However, as seen, the presence of critical fluctuations *is not a necessary condition* for the occurrence of high- T_c D -pairing. In electron systems of solids hosting flat bands, this solution can exist far from the critical line $T_N(x)$, even if the $e - e$ interaction in the Cooper channel is repulsive.

A remaining pair of solutions, corresponding to $\alpha = \pi/2$ and $\alpha = 3\pi/2$ and having $\zeta_4 = i\zeta_3 = -\zeta_2 = -i\zeta_1$, describes P -pairing [32], with the same critical temperatures given by $T_c = h_2 - h_0$.

In summary, we have generalized the Landau quasi-particle approach to determine observable properties of

systems possessing flat bands. Specifically, we have motivated and introduced a set of two Landau-like equations that replace the basic equation of Fermi liquid theory connecting the single-particle spectrum and quasiparticle momentum distribution. Analyzing a typical phase diagram of the LCCO family of electron-doped high- T_c compounds, we have demonstrated that gross properties of strange metals are incisively and economically interpreted within the flat-band scenario developed here. Importantly, we have shown that in a quadratic Brillouin zone, the BCS gap equation has nontrivial solutions even if the interaction between quasiparticles in the Cooper channel is of repulsive character. The successful description of key features of the phase diagrams and other properties of high- T_c materials should provide ample incentive for a change of theoretical course in the search for deeper understanding of non-Fermi-liquid phenomena as well as the mechanism of high- T_c superconductivity.

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